MATH 112 - Calculus II

Lab 8: The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

The harmonic series, shown above, is one of the most important series in this chapter. We have shown that even though the sequence $a_n = \frac{1}{n}$ converges to 0, the series diverges. This implies that even though the terms get infinitely small, the sum is infinitely large. For this lab we will look at this series more closely.

1. One way to show that the harmonic series diverges is believed to have first been used by J. Bernoulli. It uses groups of terms to show that the sum went to infinity. The idea is as follows:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots \frac{1}{16}}_{>\frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots \frac{1}{32}}_{>\frac{1}{2}} + \cdots$$

   Write a short explanation of how this idea "proves" that the series diverges.

2. It can be shown that:

$$\ln(n+1) \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leq 1 + \ln(n)$$

   (a) Use this to find M such that:

$$\sum_{n=1}^{M} \frac{1}{n} > 50$$

   (b) Use this to show that the sum of the first million terms is less than 15.

3. It can also be shown that:

$$\ln\left(\frac{2r+1}{r}\right) \leq \frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{2r} \leq \ln\left(\frac{2r}{r-1}\right)$$

(a) Suppose that we choose $r = 10$. Write out what the inequality above implies.

(b) Suppose that we choose $r = 100$. Write out what the inequality above implies.

(c) Find the limit as r approaches infinity of the two extremes of this inequality.

(d) What does this imply about the following limit?

$$\lim_{r \to \infty} \sum_{n=r}^{2r} \frac{1}{n}$$

(e) Explain briefly how this helps show that the harmonic series diverges.